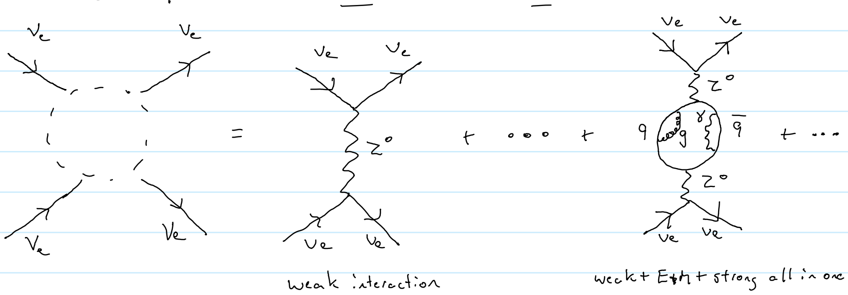
Heirarchy of interactivity : neutrinos - Weak
charged leptons - E&M, Weak
quarks - QCD, E&M, Weak

"direct" interactions but...

In a given process $\mathcal{M}$ <u>must</u> include <u>all</u> contributions!



Virtual states can bring in additional interactions!

Our next step will be to discuss QCD which means <u>quarks</u>, but we must also understand how QED plays out for quarks. We'll save the Weak interactions for last!

Quarks + QED

Quarks can replace $e^\pm, \mu^\pm, \tau^\pm$ in any QED diagram w/ appropriate charge of $g_e = e\sqrt{\frac{4\pi}{\hbar c}}$ (vertex factors).

$q = \frac{1}{3}e$ or $\frac{2}{3}e$

Everything else same: $\underbrace{u, v, \bar{u}, \bar{v}}_{\text{spin-}\frac{1}{2}\text{ quarks}}, \underbrace{\epsilon_\mu, \epsilon_\mu^*}_{\gamma\text{'s}}$ external states

propagators same, etc. (Note: color just along for ride)

Quarks + QCD

$$\mathcal{L} = \sum_{i=1}^{6} \hbar c \bar\psi_i \gamma^\mu D_\mu \psi_i + m_i c^2 \bar\psi_i \psi_i + \underbrace{\frac{1}{16\pi}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})}_{\text{usual kinetic term}}$$

$$\left.\begin{aligned} &- \frac{g}{16\pi} f^{ade} A_\mu^d A_\nu^e (\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) \\ &- \frac{g}{16\pi} f^{abc} A^{\mu b} A^{\nu c} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) \\ &+ \frac{g^2}{16\pi} f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} \end{aligned}\right\} \text{These are gluon-gluon interactions!}$$

$D_\mu \psi_i = \partial_\mu \psi_i + i g\, \lambda \cdot A_\mu \psi_i$

λ: 8 generators of $SU(3)$

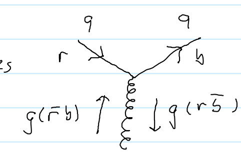
$\psi_i = \begin{pmatrix} \psi_r \\ \psi_b \\ \psi_g \end{pmatrix}_i$

Color $\underbrace{(r, g, b)}_{\text{3 charges instead of 1 in QED!}}$ plays role of "charges" and interactions via $\underbrace{\text{8 gluons}}$.

Note in QED only one γ!

One practical complication is that QCD is defined in terms of quarks, but we only observe (and experiment w/) hadrons.

A useful way to think about the strong interactions is w/ vertices. In particular, it helps to think of the gluons as "bi-colored" which could be expected since λ secretly carries 2 color indices λ_{ij}. However the gluons will always carry color-anticolor. Actually it's a bit more complicated than this!



In fact since we often work with mesons ($q\bar{q}$) and gluons ($c\bar{c}$), it is useful to work with a linearly independent set of $c\bar{c}$ basis states:

"Octet"

$|1\rangle = \frac{1}{\sqrt{2}}(r\bar{b} + b\bar{r})$ $\quad |5\rangle = -\frac{i}{\sqrt{2}}(r\bar{g} - g\bar{r})$

$|2\rangle = -\frac{i}{\sqrt{2}}(r\bar{b} - b\bar{r})$ $\quad |6\rangle = \frac{1}{\sqrt{2}}(b\bar{g} + g\bar{b})$

$|3\rangle = \frac{1}{\sqrt{2}}(r\bar{r} - b\bar{b})$ $\quad |7\rangle = -\frac{i}{\sqrt{2}}(b\bar{g} - g\bar{b})$

$|4\rangle = \frac{1}{\sqrt{2}}(r\bar{g} + g\bar{r})$ $\quad |8\rangle = \frac{1}{\sqrt{6}}(r\bar{r} + b\bar{b} - 2g\bar{g})$

Rotated into each other under $SU(3)$

Note: $|3\rangle, |8\rangle$ are "colorless" but nontrivial!

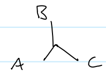
"Singlet" $\underbrace{|9\rangle = \frac{1}{\sqrt{3}}(r\bar{r} + b\bar{b} + g\bar{g})}_{\text{Invariant under } SU(3)}$

Quarks must exist in singlet combinations (for mesons).

Gluons only appear in Octet combinations. If a $|9\rangle$ gluon existed it could exist on its own and would behave much like γ. But we have never seen such a particle! Also w/ $|9\rangle$ $\underset{\text{8-generators}}{SU(3)} \rightarrow \underset{\text{9-generators}}{U(3)}$.

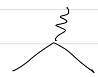
Let's compare:

ABC
- External State Labels: none
- Internal Propagators: $\dfrac{i}{q^2 - m^2c^2}$
- Vertex Factors:



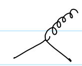
$-ig$

QED
- External State Labels: $u, \bar{u}, v\ \bar{v}$ "matter"; Internal Propagators: $\dfrac{i(\not{q} + mc)}{q^2 - m^2c^2}$
- External State Labels: $\epsilon_\mu, \epsilon^*{}_\mu$ "photons"; Internal Propagators: $-\dfrac{i\eta_{\mu\nu}}{q^2}$
- Vertex Factors:

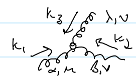


$ige\gamma^\mu$

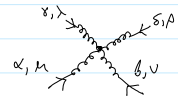
QCD
- External State Labels: $uc, \bar{u}c^\dagger, vc, \bar{v}c^\dagger$ "matter"; Internal Propagators: $\dfrac{i(\not{q} + mc)}{q^2 - m^2c^2}$
- External State Labels: $\epsilon_\mu a^\alpha, \epsilon^*_\mu a^{\alpha *}$ "gluons"; Internal Propagators: $-\dfrac{i\eta_{\mu\nu}\delta^{\alpha\beta}}{q^2}$
- Vertex Factors:



$-\dfrac{ig_s}{2}\lambda^\alpha\gamma^\mu$



$-g_s f^{\alpha\beta\gamma}\left[\eta_{\mu\nu}(k_1-k_2)_\lambda + \eta_{\nu\lambda}(k_2-k_3)_\mu + \eta_{\lambda\mu}(k_3-k_1)_\nu\right]$



$$-ig_s^2\left[f^{\alpha\beta\eta}f^{\gamma\delta\eta}(\eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda}) + f^{\alpha\delta\eta}f^{\beta\gamma\eta}(\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\lambda}\eta_{\nu\rho}) + f^{\alpha\gamma\eta}f^{\delta\beta\eta}(\eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\rho})\right]$$

Okay, so what are $c, a^\alpha, \lambda^\alpha, f^{\alpha\beta\eta}$?

c: There are 3 colors so "charge space" is 3D w/

$c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} = \text{red} \quad c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} = \text{blue} \quad c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} = \text{green}$ basis in H^3 space of color states.

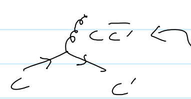
An arbitrary state can have $c = a_i c_i$

complex coefficients, hence $c^\dagger = c^{T*}$

a^α: There are 8 gluons w/

$a^1 = \begin{pmatrix}1\\0\\0\\ \vdots\end{pmatrix} \quad a^2 = \begin{pmatrix}0\\1\\0\\ \vdots\end{pmatrix} \quad \cdots \quad a^8 = \begin{pmatrix}\vdots\\0\\1\end{pmatrix}$ basis in H^8 space of gluon states.

Recall



naively $3\times 3 = 9$ gluons, but the theory only includes 8 transformed into each other by $SU(3)$ (which has 8 generators!)

The "singlet" gluon $\frac{1}{\sqrt{3}}(r\bar{r} + b\bar{b} + g\bar{g})$ does not exist (would look a lot like γ, and we haven't seen it!). Of course we haven't seen the other gluons, but for those we can use the confinement argument.

If the singlet did exist, then we would say QCD $\longleftrightarrow$ $U(3)$ (instead of $SU(3)$).

λ^α: For QED the γ^μ_{ab} matrices linked "spin-space" to "space-time", i.e. 4 4×4 matrices.

The λ^α_{ij} matrices of QCD link H^3 of "color-space" to H^8 of "gluon-space", i.e. 8 3×3 matrices (8.34).

Just like w/ γ^μ, we will leave off color-space labels and write λ^α (and c instead of c_i).

$f^{\alpha\beta\eta}$: $[\lambda^\alpha, \lambda^\beta] = 2i\underbrace{f^{\alpha\beta\gamma}}\lambda^\gamma$ 5.12 structure constants of $SU(3)$ Lie Algebra

One huge technical complication that arises is for diagrams involving internal loops. In this case we have to be very careful not to count gauge equivalent (physically indistinct) configurations more than once. To get the counting right w/out losing gauge invariance, we introduce Faddeev-Popov ghosts which are additional nonphysical fields whose sole purpose is to cancel the nonphysical gauge equivalent fluctuations of the physical fields.

Now that we have the Feynman rules for QCD down, we can use them to investigate aspects of QCD:

- Why do we find only color singlet bound states (mesons, baryons)?
- If the strong force is so strong, why is it swamped by EM at large distances?

Okay, so let's investigate why hadrons must be color singlets.

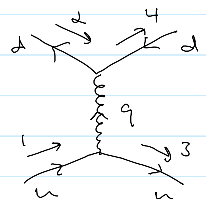
<u>Example: Mesons</u> $(q_1 \bar{q}_2)$

The idea is to see if the strong force is attractive (hence bound states) or repulsive (no bound states).
The answer we expect is:

a) For colorless singlet combination $\rightarrow$ attractive

b) For colorful octet combination $\rightarrow$ repulsive

At leading order:



$u + \bar{d} \rightarrow u + \bar{d}$

<u>Not</u> matrices in spin-space!

$$\mathcal{M} = i\,\bar{u}(3)c(3)^\dagger\left(-i\frac{g_s}{2}\lambda^\alpha\gamma^\mu\right)u(1)c(1)\left[-\frac{i\eta_{\mu\nu}\delta^{\alpha\beta}}{q^2}\right]\bar{v}(2)c(2)^\dagger\left(-i\frac{g_s}{2}\lambda^\beta\gamma^\nu\right)v(4)c(4)$$

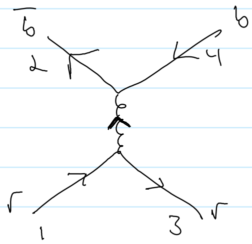
after cancelling $(2\pi)^4\delta^4(p_{tot\,in} - p_{tot\,out})$ and $\times i$:

$$= \underbrace{-\frac{g_s^2\eta_{\mu\nu}}{q^2}\bar{u}(3)\gamma^\mu u(1)\bar{v}(2)\gamma^\nu v(4)}_{\text{Exactly what we would have gotten from QED w/ } g_e \rightarrow g_s \text{, for example } e^+ + \mu^- \rightarrow e^+ + \mu^- \text{, which is attractive!!}}\ \underbrace{\left[\underbrace{(c(3)^\dagger\lambda^\alpha c(1))(c(2)^\dagger\lambda^\alpha c(4))}_{\text{color sandwiches (giving numbers)}}\right]\times\frac{1}{4}}_{\text{color ``factor''}}$$

In a sense you can model the interaction w/ a potential $V(r)$ and we are calculating $\langle \psi_f | V(r) | \psi_i \rangle$.
But we know that attractive vs. repulsive is determined by the sign of $V(r)$.

So the new step is evaluating the color factor f:

If the $u\bar{d}$ were in a colorful "octet" state, e.g. $r\bar{b} = \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle)$ then:



$$\Rightarrow \quad c(1) = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad c(2) = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad c(3) = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad c(4) = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

In the above we must have the color assignments due to overall color conservation. So in this case the gluon does not change the color at each vertex, e.g. it could be $|8\rangle = \frac{1}{\sqrt{6}}(r\bar{r} + b\bar{b} - 2g\bar{g})$.
Gluons can change color but do not have to.

Then: $$f = \frac{1}{4}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^\alpha\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\end{pmatrix}\lambda^\alpha\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{1}{4}\lambda^\alpha_{11}\lambda^\alpha_{22} = \frac{1}{4}\sum_{\alpha=1}^{8}\lambda^\alpha_{11}\lambda^\alpha_{22}$$

$$\lambda^1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} \quad \lambda^2 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix} \quad \lambda^3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} \quad \lambda^4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$$

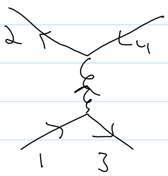
$$\lambda^5 = \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix} \quad \lambda^6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix} \quad \lambda^7 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

Finally: $$f = \frac{1}{4}\left(0 + 0 - 1 + 0 + 0 + 0 + 0 + \frac{1}{3}\right) = -\frac{1}{6}$$

Recall that the only difference between this amplitude and the attractive e+m amplitude is this color factor. Since it is negative this implies a repulsion between a quark and anti-quark in a color octet state.

If we used the colorless singlet $|9\rangle = \underbrace{\frac{1}{\sqrt{3}}(r\bar{r} + b\bar{b} + g\bar{g})}$ then:

Note: This is $\underline{\underline{\text{not}}}$ $(r+b+g)(\bar{r}+\bar{b}+\bar{g})$



$$\Rightarrow \begin{cases} c(1)\,c(2) = \frac{1}{\sqrt{3}}\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right] \\ c(3)\,c(4) = \frac{1}{\sqrt{3}}\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right] \end{cases}$$

Then $f = \frac{1}{4}(c_3^\dagger \lambda^\alpha c_1)(c_2^\dagger \lambda^\alpha c_4) = \frac{1}{4}\frac{1}{\sqrt{3}}\, c_3^\dagger \lambda^\alpha \left[\begin{pmatrix}1\\0\\0\end{pmatrix}(1\,0\,0) + \begin{pmatrix}0\\1\\0\end{pmatrix}(0\,1\,0) + \begin{pmatrix}0\\0\\1\end{pmatrix}(0\,0\,1)\right]\lambda^\alpha c_4$

$$= \frac{1}{4}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\Big[\,(1\,0\,0)\lambda^\alpha\begin{pmatrix}1\\0\\0\end{pmatrix}(1\,0\,0)\lambda^\alpha\begin{pmatrix}1\\0\\0\end{pmatrix} + (0\,1\,0)\lambda^\alpha\begin{pmatrix}1\\0\\0\end{pmatrix}(1\,0\,0)\lambda^\alpha\begin{pmatrix}0\\1\\0\end{pmatrix} + (0\,0\,1)\lambda^\alpha\begin{pmatrix}1\\0\\0\end{pmatrix}(1\,0\,0)\lambda^\alpha\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$+ (1\,0\,0)\lambda^\alpha\begin{pmatrix}0\\1\\0\end{pmatrix}(0\,1\,0)\lambda^\alpha\begin{pmatrix}1\\0\\0\end{pmatrix} + \cdots$$

$$\vdots \qquad \Big]$$

$$= \frac{1}{12}\lambda^\alpha_{ij}\lambda^\alpha_{ji} = \frac{1}{12}\underbrace{\mathrm{Tr}(\lambda^\alpha\lambda^\alpha)}_{16} = \frac{4}{3}$$

Notice that for the singlet configuration the color factor is positive and hence the strong interaction between a quark and anti-quark in this state is attractive!